Pionic pair condensation in finite isospin chemical potential

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Abstract. We study the character change of the pionic condensation at finite isospin chemical potential $\mu_{\rm I}$ by adopting the linear sigma model as a non-local interaction between quarks. At low $|\mu_{\rm I}|$ the condensation is purely bosonic, then the Cooper pairing around the Fermi surface grows gradually as $|\mu_{\rm I}|$ increases.

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Recent progress in computer power makes it possible to reliably simulate quantum chromodynamics (QCD) at finite temperature T. As for finite density, however, the well known sign problem limits simulations. Alternatively, QCD at finite isospin chemical potential $\mu_{\rm I} = \mu_u - \mu_d$ (where μ_u and μ_d denoting the chemical potential of u and d quark, respectively) as well as the SU(2) color systems, in which the sign problem does not exist, are studied to give insights into the actual finite density physics. One of the most interesting aspects of the finite $\mu_{\rm I}$ systems is that they accommodate pion condensation for $|\mu_{\rm I}| > m_\pi$ [1], with m_π denoting the mass of pions. Son and Stephanov [2] predicted that the pion condensed phase evolves to Cooper pairing between u and $\bar{d}$ (d and $\bar{u}$) for $\mu_{\rm I} > 0$ (< 0) at high $|\mu_{\rm I}|$, but the quantitative process of the character change of the condensation has not been discussed.

The BEC–BCS crossover has long been expected to occur in various quantum systems [3, 4]; it was experimentally observed in ultra cold atomic gases, in which the strength of the interaction can be tuned artificially, only recently. At least in principle, it can occur also in systems governed by the strong interaction, in which the strength of the interaction can not be tuned artificially. Rather, the change in the environment, typically density, would lead to the crossover [5]. In symmetric nuclear matter, the neutron (n)–proton (p) pairing in the $^3S_1$ + $^3D_1$ channel that leads to bound deuteron formation was studied [6]. The n–n and p–p $^1S_0$ pairing, that has attracted attention from viewpoints of both nuclear structure and neutron stars, however, does not reach the BEC [7, 8]. In intermediate density quark (q) matter, the present author discussed that the spatial extension of quark Cooper pairs in a color superconductor is comparable with the mean interparticle distance [9]. Later, a wide enough density region was studied [10] and showed that the diquark pairing becomes weak at very high density.

Since the mechanism of the fermion-antifermion condensation that produces the fermion mass is essentially the same as the BCS pairing as recognized in Nambu and Jona-Lasinio's celebrated paper [11], the evolution of the charged pion condensation to q–$\bar{q}$ Cooper pairs can be analyzed in the context of the BEC–BCS crossover in terms of the spatial structure of the pion condensation. To this end, one must introduce a non-

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local interaction between q and $\bar{q}$ that gives momentum dependent condensations. In the present study, we adopt the linear sigma model, which respects chiral symmetry, as an inter-quark interaction, since 1) the pion condensation occurs as a spontaneous symmetry breaking among three pions that have light but non-zero masses after the chiral symmetry breaking between the sigma meson and the pions, and 2) the effect of high $|\mu_I|$ on it has long been studied [1, 12, 13].

The adopted effective Lagrangian for the quarks, sigma mesons and pions is

$$\begin{aligned}
\mathscr{L}_{\rm eff} &= \mathscr{L}_{\rm q} + \mathscr{L}_{\rm M} + \mathscr{L}_{\rm couple},\\
\mathscr{L}_{\rm q} &= \bar{q}(i\partial\!\!\!/ - m_q + \frac{\mu_{\rm I}}{2}\gamma^0\tau_3)q,\\
\mathscr{L}_{\rm M} &= \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}) - U(\sigma,\vec{\pi}) + \mu_{\rm I}(\pi_1\dot{\pi}_2 - \pi_2\dot{\pi}_1) + \frac{\mu_{\rm I}^2}{2}(\pi_1^2+\pi_2^2),\\
U(\sigma,\vec{\pi}) &= \frac{\lambda^2}{4}(\sigma^2+\vec{\pi}^2)^2 - \frac{\lambda^2 f_\pi^2 - m_\pi^2}{2}(\sigma^2+\vec{\pi}^2) - f_\pi m_\pi^2\sigma,\\
\mathscr{L}_{\rm couple} &= -G\bar{q}(\sigma + i\gamma^5\vec{\tau}\cdot\vec{\pi})q, \qquad (1)
\end{aligned}$$

where f_π and m_π stand for the pion decay constant and the pion mass, respectively. Hereafter, quantum fluctuations are indicated by primes.

It is well known that, in the mean field level, $U_{\rm eff} = U(\sigma,\vec{\pi}) - \frac{\mu_{\rm I}^2}{2}(\pi_1^2+\pi_2^2)$ has the minimum at

$$\langle\sigma\rangle = \frac{f_\pi m_\pi^2}{\mu_{\rm I}^2}, \quad \langle\pi\rangle^2 = \frac{\mu_{\rm I}^2 - m_\pi^2}{\lambda^2} + f_\pi^2 - \langle\sigma\rangle^2 \qquad (2)$$

for $|\mu_{\rm I}| > m_\pi$, assuming $\langle\pi_3\rangle = 0$ [1, 12]. We take $\langle\pi_1\rangle = \langle\pi\rangle$ and $\langle\pi_2\rangle = 0$ without loss of generality. After expanding $\mathscr{L}_{\rm M}$ up to the quadratic terms in σ' and π_i', diagonalization of the coupled Klein-Gordon equations for σ', π_1' and π_2' gives the mass eigenvalues, one of which is zero as done in Ref. [12]. But the meson mixing can not be calculated since the 3×3 mass matrix is not regular. Thus, another approximation must be sought. Since the essential character of the meson propagation in the pion condensed phase is the rotational motion in the isospin space, we adopt a polar coordinate representation,

$$\pi_\pm = \frac{1}{\sqrt{2}}(\pi_1 \pm i\pi_2) = \frac{1}{\sqrt{2}}\pi\exp(\pm i\theta) = \frac{1}{\sqrt{2}}(\langle\pi\rangle + \pi')\exp(\pm i\theta), \qquad (3)$$

without expanding the angular field. This representation assures the conservation of the third component of the isospin current of the total system, within the quadratic terms of the fluctuating quantum fields. After confirming this point, we write down the coupled Klein-Gordon equations retaining the lowest order terms in each equation as

$$\begin{aligned}
&\partial_\mu\partial^\mu\sigma' + (2\lambda^2\langle\sigma\rangle^2 + \mu_{\rm I}^2)\sigma' + 2\lambda^2\langle\sigma\rangle\langle\pi\rangle\pi' = -G(\bar{q}q)',\\
&\partial_\mu\partial^\mu\pi' + 2\lambda^2\langle\pi\rangle^2\pi' + 2\lambda^2\langle\sigma\rangle\langle\pi\rangle\sigma' - 2\mu_{\rm I}\langle\pi\rangle\dot{\theta} = -G(\bar{q}i\gamma^5\tau_1 q)',\\
&\langle\pi\rangle\partial_\mu\partial^\mu\theta = -G(\bar{q}i\gamma^5\tau_2 q)', \quad \partial_\mu\partial^\mu\pi_3' + \mu_{\rm I}^2\pi_3' = -G(\bar{q}i\gamma^5\tau_3 q)'. \qquad (4)
\end{aligned}$$

Here we make one additional approximation to handle the set of equations: We ignore $-2\mu_{\rm I}\langle\pi\rangle\dot{\theta}$ in the second equation that corresponds to the Coriolis coupling. The obtained set contains 1) the σ–π mixing, and 2) the rotational massless field due to the existence of the pion condensation $\langle\pi\rangle$.

The equation of motion of the quark propagator, $G^{ij}_{\alpha\beta}(x-x')$ where i, j and α, β represent isospin and Dirac indices respectively, is given by

$$(i\partial\!\!\!/ - m_q + \frac{\mu_{\rm I}}{2}\gamma^0\tau_3)G(x-x') = \delta^4(x-x') - iG\langle\tilde{0}|T(\sigma(x)+i\gamma^5\overrightarrow{\tau}\cdot\overrightarrow{\pi}(x))q(x)\bar{q}(x')|\tilde{0}\rangle. \tag{5}$$

After sorting the mean field terms in $\sigma + i\gamma^5\overrightarrow{\tau}\cdot\overrightarrow{\pi}$ to the left-hand side, we substitute the inverted Eq.(4) to Eq.(5). Then we perform the Wick decomposition of the 4-point term. Only the Fock terms that lead to the non-local selfenergy $\Sigma(x-y)$, which depends on $G(x-y)$, appear since the Hartree (mean field) terms have already been sorted. By a Fourier transformation and an isospin decomposition, we obtain a Gor'kov [14] type equation,

$$\begin{pmatrix} \gamma^0(\omega - h \pm \mu_{\rm I}/2) + \Sigma^0 \pm \Sigma^3 & -G\langle\pi\rangle i\gamma^5 + \sqrt{2}\Sigma^{\mp} \\ -G\langle\pi\rangle i\gamma^5 + \sqrt{2}\Sigma^{\pm} & \gamma^0(\omega - h \mp \mu_{\rm I}/2) + \Sigma^0 \mp \Sigma^3 \end{pmatrix} \begin{pmatrix} G^0 \pm G^3 \\ \sqrt{2}G^{\pm} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6}$$

with $h = \gamma^0\gamma\cdot\mathbf{k} + \gamma^0(m_q + G\langle\sigma\rangle)$ being the free single particle Hamiltonian with the constituent quark mass, $M_q = m_q + G\langle\sigma\rangle$. This form indicates that the present subject is a pairing problem. In the following we take the lower one of the double sign. Expanding $G_{\alpha\beta}(k)$ by the plane wave spinor and taking the residue at the quasiparticle pole, finally we obtain a 4×4 hermitian matrix equation at each k,

$$\begin{pmatrix} e - E_k^+ - m_1 & -\sigma_1 & -\pi & -\delta \\ -\sigma_1 & e + E_k^- - \tilde{m}_1 & -\tilde{\delta} & -\tilde{\pi} \\ -\pi & -\tilde{\delta} & e + E_k^+ - \tilde{m}_2 & -\sigma_2 \\ -\delta & -\tilde{\pi} & -\sigma_2 & e - E_k^- - m_2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0, \tag{7}$$

with $E_k^{\pm} = E_k \pm \mu_{\rm I}/2$. Here the (real) Bogoliubov amplitudes are defined as

$$A = \langle\tilde{0}|a_d\eta^\dagger|\tilde{0}\rangle, \quad B = \langle\tilde{0}|b^\dagger_{-d}\eta^\dagger|\tilde{0}\rangle, \quad C = -i\langle\tilde{0}|b^\dagger_{-u}\eta^\dagger|\tilde{0}\rangle, \quad D = -i\langle\tilde{0}|a_u\eta^\dagger|\tilde{0}\rangle, \tag{8}$$

with $\eta^\dagger$ denoting the creation operator of the eigen quasiparticle. This type of equation appears also in the cases of the relativistic 1 flavor pairing including the Dirac sea [15] and the non-relativistic 2 flavor pairing [16]. In Eq.(7) 10 kinds of mass and gap functions are independent. Among them, the momentum dependent gap for the quark is

$$\pi(k) = -i\bar{U}(k)(-G\langle\pi\rangle i\gamma^5 + \sqrt{2}\Sigma^+)V(k), \tag{9}$$

here the first term stems from the momentum independent pion condensation $\langle\pi\rangle$ of the meson system, and the second one from the non-local Fock selfenergy that is a function of $A(k')$–$D(k')$. Therefore the equations for all momenta are coupled. Solving them selfconsistently determines all the physical quantities: The Bogoliubov amplitudes, quasiparticle energies, and the mass and gap functions at each $\mu_{\rm I}$. Then the pair wave functions and the coherence length are calculated from them.

Now we proceed to numerical calculations. Parameters used are the current quark mass $m_{\rm q} = 0.0055$ GeV, the momentum cutoff $\Lambda = 0.63$ GeV, the pion decay constant

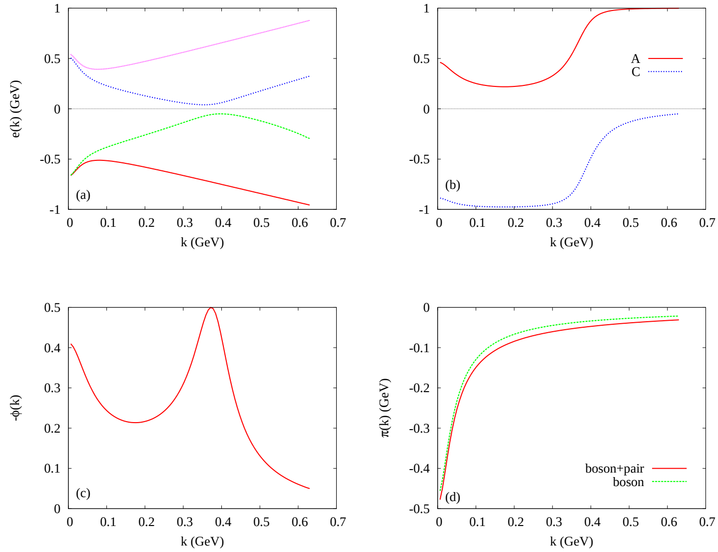

FIGURE 1. Momentum dependence of various quantities at $\mu_{\rm I}$ = -0.5 GeV: (a) the quasiparticle energies, (b) the Bogoliubov amplitudes, (c) the pair wave function, and (d) the gap function. Note that (b) – (d) are associated with the third (from the bottom) solution in (a).

$f_\pi = 0.093$ GeV, the pion mass $m_\pi = 0.138$ GeV, the parameter in the linear sigma model $\lambda = 4.5$, and the quark–meson coupling $G = 3.3$. Calculations are done for $\mu_{\rm I} < 0$.

Figure 1 shows the results at $|\mu_{\rm I}| = 0.5$ GeV $\gg m_\pi$. Figure 1 (a) is the quasiparticle energy diagram as a function of the relative momentum k (dispersion relation). Its unperturbed structure is quite simple: The positive and negative energy u (d) quark levels with $\pm E_k$ are shifted upward (downward) by $|\mu_{\rm I}|/2$. Then, the negative energy u, that is the hole state of $\bar{u}$, and the positive energy d interact around the Fermi surface. This means the $d\bar{u}$ pairing. Hereafter we name these quasiparticle (hole) levels the first, second, third and fourth, from the bottom. In the following discussion, we concentrate on the third level, the lower quasiparticle. In the case of this lower quasiparticle, the selfconsistently determined solution consists only of A and C corresponding to u and v in the usual notation, that is, other amplitudes B and D are zero at each k. Figure 1 (b) shows the Bogoliubov amplitudes A and C. Aside from the bump around $k = 0$ mentioned below, the hole character changes gradually to the particle character around the Fermi surface as the usual Cooper pairing. This leads to the peak in the pair wave function $\phi(k) = A(k)C(k)$ shown in Fig. 1 (c). The bump around $k = 0$ is a novel feature of the present case; this is brought about by the mesonic contribution $\langle \pi \rangle$ to the gap function $\pi(k)$ (see Eq.(9)) as shown in Fig. 1 (d). In this gap function, the mesonic and the Cooper pair components are comparable around the Fermi surface, whereas the former is dominant around $k = 0$ because of the k dependence $\propto M_q/E_k$.

Figure 2 shows the $\mu_{\rm I}$ dependence of the pair wave function. Figure 2 (a) shows the pair wave functions at several $\mu_{\rm I}$s as functions of the momentum. This shows that the

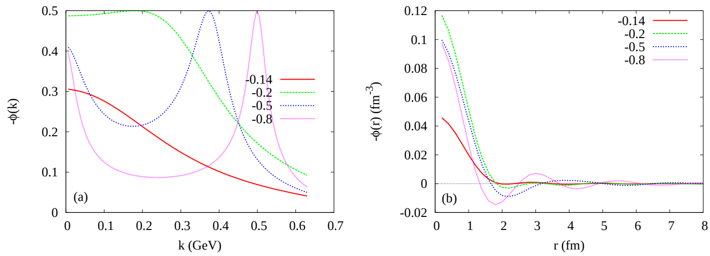

FIGURE 2. Isospin chemical potential dependence of the pair wave function: (a) in the *k* space, and (b) in the *r* space.

peak due to the Cooper pairing can not be seen at low $|\mu_\mathrm{I}|$. Actually, q and $\bar{q}$ are bound to each other for $|\mu_\mathrm{I}| < 2M_q$. Thus, we can conclude that the pionic condensation has a mixed character: Purely bosonic just after the appearance of the condensation, then the Cooper pairing gradually grows as $|\mu_\mathrm{I}|$ increases with retaining significant bosonic component. To look into the spatial structure of Cooper pairs more closely, we Fourier transform $\phi(k)$. The results for several μ_Is are shown in Fig. 2 (b) as functions of the relative distance. Obviously those for higher $|\mu_\mathrm{I}|$ wave till longer distance. The coherence length, indicating the spatial extension of the q-$\bar{q}$ pairs, increases from 0.7 fm at $|\mu_\mathrm{I}| = 0.14$ GeV to 3.2 fm at $|\mu_\mathrm{I}| = 0.8$ GeV.

To summarize, we have studied the momentum dependence of the pionic gap function $\pi(k)$ by adopting the linear sigma model as an inter-quark interaction at finite isospin chemical potential and zero temperature. The present framework is useful for the cases that can not be represented as a usual gap equation, including those in which there are more than one gap. The character of the condensation is bosonic at low $|\mu_\mathrm{I}|$, then the Cooper pairing gradually grows as $|\mu_\mathrm{I}|$ increases.

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