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Two-dimensional SU(N) gauge theory on the light-cone

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Abstract

Two-dimensional SU(N) gauge theory is accurately analyzed with the light-front Tamm-Dancoff approximation, both numerically and analytically. The light-front Einstein-Schrödinger equation for mesonic mass is reduced to 't Hooft equation in the large N limit with g^2N fixed, where g is the coupling constant. Two mesonic and one baryonic bound states are obtained numerically in the strong coupling region, $g^2N \gg m^2$ for small N, where m is the bare quark mass, and the N- and m-dependences of the hadronic masses are shown. The lightest meson and the baryon consist predominantly of valence quarkes. The second mesonic state is highly relativistic in the sense that it has a large four body component in addition to the valence one. Our results are consistent with results of the lattice calculation for SU(2) and also with the prediction of bosonization for ratios of the two mesonic masses to the baryonic one in the strong coupling limit.

1 Introduction and summary

Two dimensional SU(N) quantum chromodynamics QCD$(N)_2$ is a good model studying ideas and tools which are expected to be feasible in analyses of QCD in $3+1$ dimensions. 't Hooft introduced the model to test the power of the $1/N$ expansion[1]. He summed planer diagrams which dominate the leading order in the expansion and derived an equation. The 't Hooft equation is valid in the large N limit with g^2N fixed, where g is the coupling constant. The mass spectram of the equation reveals a nearly straight "Regge tragectory".

The large N limit corresponds to the weak coupling one. The $1/N$ expansion then works in the weak coupling regime, but not in the strong coupling one, because it is almost impossible to calculate higher-order terms in the expansion. For this reason, QCD$(N)_2$ in the strong coupling regime has been studied with some other methods so far. Neverthless, the dynamics is not understood well in the region.

Recently the light-front Tamm-Dancoff(LFTD) approximation[6] has been proposed as one of the alternative non-perterbative tools to lattice gauge theory. In the standard equal-time field theory the vacuum state is an infinite sea of constituents and hadrons arise as excitations of the sea. It is then unlikely that the vacuum and the hadrons are well described with a finite number of constituents. In fact such a truncation of the Fock space, i.e., the Tamm-Dancoff approximation[7], causes some serious problem[6]. Such problem do not appear in light-front field theory[8], owing to the fact that the vacuum is trivial on the light cone[6].

In this work, QCD$(N)_2$ is investigated in the region, $g^2N \gg m^2$ with LFTD; the region corresponds to the strong coupling region for small N, and for large N it covers not only the strong coupling region but also the medium and weak coupling ones. We first derive the light-cone Hamiltonian , P^-, and the Einstein-Scrö dinger(ES) equation, $2\mathcal{P}P^-|\Psi\rangle = M^2|\Psi\rangle$, for hadronic mass and wave function, M and $|\Psi\rangle$, in the framework of the light-front field theory[8]. As the Tamm-Dancoff approximation, the mesonic SU(N) wave function is described with two-body and four-body states, and the baryonic wavefunction with the N-body state. Inclusion of the four-body state is essential to obtain the second lightest relativistic bound states. The ES equation is numerically solved by diagonalyzing P^- within the space spanned by a finite number of basis functions. All tools needed for this calculation are prepared by our previous work[10] for the massive Schwinger model. Only the mesonic case is presented in this manuscript. The details are written in Ref. [11].

2 Light-front Tamm-Dancoff approximation

2.1 Light-cone Hamiltonian

The Lagrangian density of QCD$(N)_2$ for interacting quark and gauge fields, ψ and A^a_μ $(a = 1, 2, \ldots, N^2-1)$, is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{1}$$

where $D_\mu = \partial_\mu - igA^a_\mu T^a$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu$ for the generator T^a and the structure constant f_{abc} of SU(N). Light-front field theory[8] starts with the introduction of light-cone coordinates, $x^\mu = (x^+, x^-) \equiv ((x^0+x^1)/\sqrt{2}, (x^0-x^1)/\sqrt{2})$; for any other vector, $V^\pm = (V^0 \pm V^1)/\sqrt{2}$. (We take the same notations and conventions as in Ref. [10, 11].) The equations of motion are

$$i\sqrt{2}\partial_-\psi_L = m\psi_R, \tag{2}$$
$$i\sqrt{2}\partial_+\psi_R = m\psi_L - \sqrt{2}gA^-\psi_R, \tag{3}$$
$$\partial_-^2 A^{a-} = \sqrt{2}g\psi_R{}^\dagger T^a\psi_R, \tag{4}$$
$$-\partial_-\partial_+A^{a-} = \sqrt{2}g\psi_L{}^\dagger T^a\psi_L + gf_{abc}A^{b-}\partial_-A^{c-}. \tag{5}$$

for the light-cone gauge, $A^{a+}=0$, where $\psi=(\psi_R,\psi_L)^T$. The first and third equations do not involve the time derivative (∂_+) and are therefore just constraints which determine ψ_L and A^- in terms of ψ_R. Thus, ψ_L and A^- are not independent variables and not subject to a quantization condition. The constraints are then solved with the inverse derivative operator ∂_-^{-1},

$$\psi_L(x^-) = -i\frac{m}{2\sqrt{2}}\int dy^-\epsilon(x^- - y^-)\psi_R(y^-), \quad (6)$$

$$A^{a-}(x^-) = \sqrt{2}g\frac{1}{\partial_-^2}\psi_R{}^\dagger(x^-)T^a\psi_R(x^-), \quad (7)$$

where $\epsilon(x)$ is 1 for $x>0$ and -1 for $x<0$. The only independent variable ψ_R is quantized by an anticommutation relation at the equal light-cone time $x^+=y^+$,

$$\{\psi_{iR}(x),\psi_{jR}{}^\dagger(y)\}_{x^+=y^+} = \frac{1}{\sqrt{2}}\delta_{ij}\delta(x^- - y^-). \quad (8)$$

Adopting the light-cone coordinates and ligt-cone gauge thus reduces a number of independent variables. This is an advantage of light-front field theory. The energy-momentum vectors commute mutually and are therefore constants of motion. The time component(light-cone Hamiltonian) is

$$P^- = -\frac{im^2}{2\sqrt{2}}\int dx^- dy^- \psi_R{}^\dagger(x^-)\epsilon(x^- - y^-)\psi_R(y^-) - \frac{g^2}{2}\int dx^- j^{a+}(x^-)\frac{1}{\partial_-^2}j^{a+}(x^-). \quad (9)$$

The field ψ_R is expanded at $x^+=0$ in terms of free waves [12], each with momentum k^+,

$$\psi_{iR}(x^-) = \frac{1}{2^{1/4}}\int_0^\infty \frac{dk^+}{2\pi\sqrt{k^+}}[b_i(k^+)e^{-ik^+x^-} + d_i{}^\dagger(k^+)e^{ik^+x^-}], \quad (10)$$

with

$$\{b_i(k^+),b_j{}^\dagger(l^+)\} = \{d_i(k^+),d_j{}^\dagger(l^+)\} = 2\pi k^+\delta_{ij}\delta(k^+ - l^+), \quad (11)$$

The color current, $j^{a+}\equiv\sqrt{2}:\psi_R{}^\dagger T^a\psi_R:$, is normal-ordered with respect to the creation and annihilation operators. The charge is then

$$Q^a = \int dx^- j^{a+} = \sum_{i,j}^N (T^a)_{ij}\int_0^\infty \frac{dk^+}{2\pi k^+}[b_i{}^\dagger(k^+)b_j(k^+) - d_j{}^\dagger(k^+)d_i(k^+)]. \quad (12)$$

The last term in P^- can be rewritten with the standard Fourier transform [13],

$$\int dx_1^- j^{a+}(x_1^-)\frac{1}{\partial_-^2}j^{a+}(x_1^-) = \frac{1}{2}\int dx_1^- dx_2^- j^{a+}(x_1^-)|x_1^- - x_2^-|j^{a+}(x_2^-) - \frac{1}{4\eta}\sum_{a=1}^{N^2-1}Q^aQ^a + O(\eta), \quad (13)$$

where an integral form of the inverse operator ∂_-^2 is used. The term $Q^2/4\eta$ ($Q^2\equiv\sum Q^aQ^a$) enforces confinement, restricting finite eigenvalues to the color-singlet ($Q^2=0$) subspace.

The Hamiltonian can be expressed with the creation and annihilation operators. The Hamiltonian does not involve any term having the creation operators only or the annihilation ones only. This indicates that the Fock vacuum is an eigenstate of P^-, i.e., a true vacuum. The property of the Hamiltonian stems from the conservation of the total light-cone momentum. Each particle must have either zero or a positive momentum, as shown in Eq. (2.8). The creation or the annihilation of particles, each with positive k^+, breaks the conservation. An exception is the zero mode ($k^+=0$): Only the mode can make the true vacuum non-trivial without breaking the conservation. The mode is thus responsible for non-trivial structure of vacua such as spontaneous symmetry breaking. In the present model, however, the mode is prohibited as long as $m\neq 0$, because the mass term in $P^-_{\rm free}$ enforces the eigenstate of P^- to vanish at $k^+=0$ [14].

2.2 Hadronic color-singlet states

The conserved color charges Q^a ($a=1,2,\ldots,N^2-1$) are generators of SU(N). These can be recombined into $N-1$ operators being mutually commutable and $N(N-1)/2$ pairs of raising and lowering operators. Whenever these operators act on color-singlet states, the value is always zero. Using the condition, one can easily construct color-singlet states of meson and baryon. The color-singlet states are expanded in terms of the number of quarks and antiquarks, and truncated to the two- and four-body components in the case of the mesonic state and to the N-body one in the case of the baryonic state. The Q^a's do not couple the truncated space with the remainder, so they keep proper commutation relations between them within the truncated space. The truncation, i.e. the Tamm-Dancoff approximation [7], thus does not break the SU(N) symmetry (see the details in Ref. [11]).

3 Numerical method and results

3.1 Basis functions

The truncated ES equation for hadron masses are numerically solved with the variational method: The wave functions are expanded in terms of basis functions, and the coefficients of expansion are determined by diagonalizing P^- in the space spanned by the basis functions. All tools needed for computations are shown in Ref. [10, 11].

3.2 Numerical results

In general, M calculated with the variational method depends on N_α which calacterize the size of the space spanned by the basis functions, unless the space is large enough to yield an accurate M. In the present calculation, the space would be sufficiently large, since the dependence is very weak, owing to the effective choice of basis functions. Hereafter, m and M are presented in units of $\sqrt{g^2N/2\pi}$.

The m-dependence of hadoron masses obtained with full-fledged calculations is shown in Fig. 1 for both SU(2) and SU(3). There are two mesonic and one baryonic bound states in the range $m < 0.1$. The lightest meson is composed predominantly of valence quarks; in the case of SU(2), for example, $P_2 = 98.3\%$ and $P_4 = 1.7\%$ at $m = 10^{-4}$, where $P_2(P_4)$ is a probability of being in the $q\bar{q}(qq\bar{q}\bar{q})$ state. The lightest mesonic state is odd under charge conjugation, because the two-body component $(q\bar{q})$ is symmetric under $x_1 \leftrightarrow x_2$. The second lightest state is, on the other hand, highly relativistic in the sense that $P_2 \sim P_4$; in the case of SU(2), for example, $P_2 = 42.9\%$ and $P_4 = 57.1\%$ at $m = 10^{-4}$. This state is even under charge conjugation, since the two-body piece is antisymmetric under $x_1 \leftrightarrow x_2$.

The lightest mesonic mass is calculated also with the bosonization[2, 4], the lattice theory [3, 4] and the DLCQ method[5]. These results are compared with ours, in Fig. 2, for SU(3). The lattice calculation has much larger errors than ours, but both are consistent with each other within the errors. The DLCQ result matches well to ours at $m > 0.3$, but it lies too low at $m < 0.2$.

The approximate solutions to M_1(lightest meson) and M_b(lightest baryon) are compared with numerical ones obtained with the full-fledged calculations, in two cases of SU(2) and SU(3). For SU(2), the apptoximate M_1 is exactly equal to the approximate M_b. They are depicted by a single dashed line in Fig. 3(a), and compared with the numerical solurions for M_1(solid line) and for M_b(dot-dashed line). The approximate solution matches well to the numerical results for both M_1 and M_b. For SU(3) in Fig. 3(b), the approximate solutions well reproduce the numerical ones, for both M_1 and M_b, at $m < 0.1$. The agreement would be seen also at N larger than 3; this is true at least for M_1 (see Fig. 4). The N-dependence of M_1 and M_b is thus obtained accutately with the accutately with the apptoximate solutions, as long as $m^2 \ll 1$.

The N-dependence of M_1 and M_b is shown at $m = 10^{-4}$, in Fig. 4 where N is varied widely from 2 to ∞. The approximate solution to M_1(dashed line) well simulates the numerical solution (solid line). As expected from the weak N-dependence of the approximate M_1, M_1 at small N is close to that at $N \to \infty$(the lightest 't Hooft mass). The second mesonic mass is below the threshold($2M_1$) for $N = 2, 3$, but not for $N \geq 4$. The second mesonic state is thus bounded only for such small N. The existence of the second bound state at small N is unpredictable from the leading order in $1/N$, since the state becomes unbound at $N \to \infty$. The $1/N$ expansion thus works well for the first mesonic mass, but not for the second mass.

4 Discussion

Some unsettled problems are discussed.

(1)A natural expectation for the Tamm-Dancoff approximation is that it works best at weak coupling rather than strong coupling. The present work, however, points out that it works in both the regions. As an evidence, the lightest meson consists only of a q-$\bar{q}$ pair in the strong coupling limit, so that the meson has small four-body components even for large but finite g. Similar results are seen in the massive Schwinger model[9, 10]. It is not obvious whether the approximation still works in four dimensions, since the four-simensional QCD Hamiltonian is much more complicated because of the transverse directions.

(2)Our truncated Fock space consists of the two- and four-body states in the mesonic case. Further inclusion of six-body states would produce the third mesonic bound state at strong coupling. This may be expected from the following considerations in the limit of strong coupling: The bosonization[2] predicts for SU(N) meson that there appear $2N - 1$ massless bound states, and DLCQ[5] does that there are many massless mesonic states and the n-th state consists of n components from two-body to $2n$-body.

Throughout this work, we conclude that LFTD is a powerful tool for computing non-perturbative quantities such as hadronic masses. We believe that LFTD is more useful than the $1/N$ expansion and the bosonization which are valid only in a particular situation such as the large N limit or the large g/m limit.

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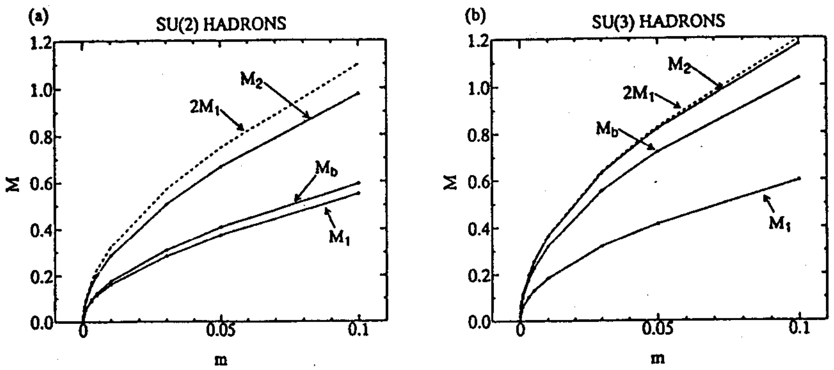


Fig. 1 Masses of the lowest two mesonic(M_1 and M_2) and one baryonic(M_b) bound states, obtained with the full-fledged calculations, are presented in units of $\sqrt{g^2N/2\pi}$ as a function of the quark mass m; (a)for SU(2) and (b)for SU(3). They are shown as points and the solid line guides the eye. The two-body decay threshold, $2M_1$ is also shown by the dashed line.

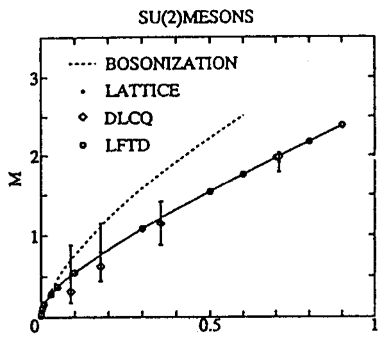
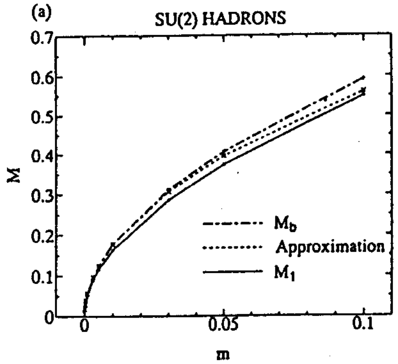


Fig. 2 Some theoretical results for a mass of the lowest SU(2) meson are presented in units of $\sqrt{g^2N/2\pi}$ as a function of m; the result of our full-fledged calculation are shown by open circles, the DLCQ result [3] by diamonds and the bosonization result[2,4] by crosses. The lattice estimates [4] are shown as points with error bars.



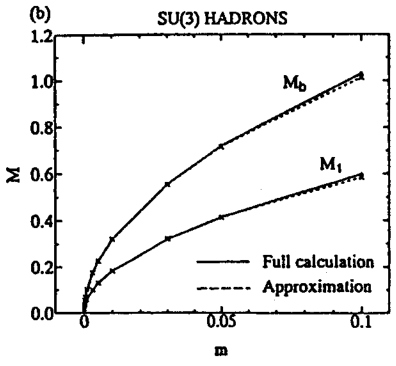


Fig. 3 Numerical and approximate masses of the lowest mesonic (M_1) and baryonic(M_b) states are shown in units of $\sqrt{g^2N/2\pi}$ as a function of the quark mass m. (a)For the SU(2) case, the numerical solutions to M_1 and M_b are graphed with the solid and dot-dashed lines, respectively, while the approximate ones to both of them are degenerate and therefore graphed with a dashed line. (b)For SU(3) case, the numerical and the approximate solutions are graphed with the solid and dashed lines, respectively, for both M_1 and M_b.

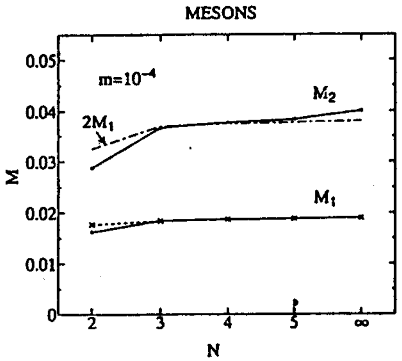


Fig. 4 Numerical solutions to the second lightest mesonic masses, calculated for SU(N) by adopting a quark mass $m = 10^{-4}$, are shown by the solid lines as a function of N in comparison to that of the lightest meson. The approximate solution, which is available only for the lightest meson, is shown by the dashed line. The two-body decay threshold, $2M_1$, is also shown by the dot-dashed line. Here all the masses are given in units of $\sqrt{g^2N/2\pi}$.