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Toshio Nakata
Department of Mathematics,
Fukuoka University of Education,
Akama-Bunkyomachi, Munakata, Fukuoka,
811-4192, JAPAN.

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11356. *Proposed by Michael Poghosyan, Yerevan State University, Yerevan, Armenia.* Prove that for nay positive integer n,

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!}.$$

Solution by Toshio Nakata, Fukuoka University of Education, Fukuoka, Japan. Using hypergeometric series, we have

$$\sum_{k=0}^{n} \frac{\binom{n}{k}^2}{(2k+1)\binom{2n}{2k}} = {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -n \\ \frac{3}{2}, -n+\frac{1}{2}\end{matrix} \,\middle|\, 1\right] = \frac{\left(\frac{3}{2}-\frac{1}{2}\right)^{\overline{n}}\left(\frac{3}{2}-\frac{1}{2}\right)^{\overline{n}}}{\left(\frac{3}{2}\right)^{\overline{n}}\left(\frac{3}{2}-\frac{1}{2}-\frac{1}{2}\right)^{\overline{n}}} = \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!},$$

where $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$. The second equality holds by Saalschürtz's identity (see R. Graham, D. Knuth, O. Patashnik, Concrete Mathematics, Addison-Wesley, Equation (5.97)).