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11369. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Prove that for all real t, and all $\alpha \geq 2$,

$$e^{\alpha t} + e^{-\alpha t} - 2 \leq (e^t + e^{-t})^\alpha - 2^\alpha$$

Solution by Toshio Nakata, Fukuoka University of Education, Fukuoka, Japan. Fix $\alpha \geq 2$. Since

$$\begin{aligned}(\text{LHS}) &= 2(\cosh \alpha t - 1) = 2\alpha \int_0^t \sinh \alpha x dx,\\ (\text{RHS}) &= (2\cosh t)^\alpha - 2^\alpha = 2\alpha \int_0^t (2\cosh x)^{\alpha-1} \sinh x dx,\end{aligned}$$

we will show

$$\int_0^t \sinh \alpha x dx \leq \int_0^t (2\cosh x)^{\alpha-1} \sinh x dx \quad \text{for all real } t.$$

Because $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$, we only check the above equation for $t > 0$. Comparing each integrand, we will show

$$\frac{\sinh \alpha x}{\sinh x} \leq (2\cosh x)^{\alpha-1} \quad \text{for } x > 0.$$

Namely,

$$\frac{e^{\alpha x} - e^{-\alpha x}}{e^x - e^{-x}} \leq (e^x + e^{-x})^{\alpha-1} \quad \text{for } x > 0.$$

Putting $e^{2x} = s$, we have

$$\frac{s^\alpha - 1}{s - 1} \leq (s+1)^{\alpha-1} \quad \text{for } s > 1. \tag{1}$$

Therefore it is sufficient to show (1). Since $\alpha \geq 2$ and $s > 1$ we have

$$(s-1)\int_s^{s+1}(\alpha-1)x^{\alpha-2}dx \geq (s-1)(\alpha-1)s^{\alpha-2} \geq \int_1^s(\alpha-1)x^{\alpha-2}dx.$$

Hence

$$(s-1)\{(s+1)^{\alpha-1} - s^{\alpha-1}\} \geq s^{\alpha-1} - 1.$$

So we obtain (1) [1].

[1] Of course, if $\alpha \geq 2$ is integer then we have

$$\sum_{k=1}^{\alpha-1} s^k \leq \sum_{k=1}^{\alpha-1} \binom{\alpha-1}{k} s^k.$$

The above equation is corresponding to (1).