The lost boarding pass problem: converse results

SHOHEI KUBO TOSHIO NAKATA and NAOKI SHIRAISHI

1. *Introduction and results*

This Article is a follow-up to a recent Gazette Article about *the lost boarding pass problem* by Grimmett and Stirzaker [1]. According to their book [2, 1.8.39, p.10], it seems that they recognised this lovely problem in 2000 or earlier. We quote it with suitable minor changes.

> **(The lost boarding pass problem)** The n passengers for a Bell-Air flight in an airplane with n seats have been told their seat numbers. They get on the plane one by one. The first person loses his or her boarding pass, and sits in a randomly chosen seat. Subsequent passengers sit in their assigned seats whenever they find them available, or otherwise in a randomly chosen empty seat.
>
> (I) Suppose that the first person sits in a seat chosen uniformly at random from n available. What is the probability that the last passenger finds his or her assigned seat to be free?
>
> (II) Suppose that the first person sits in a seat chosen uniformly at random *except* his or her assigned seat. What is the probability of the previous question?

For the time being, we assume $n \geq 2$. The solutions of (I) and (II) written in [2, 1.8.39, p.197] are

$$\frac{1}{2} \quad \text{and} \quad \frac{n-2}{2(n-1)}, \tag{1}$$

respectively. To discuss the problem we use some notation. For $l \in \{1, \ldots, n\}$ let N_l be the random seat number of the passenger l, so that $(N_1, \ldots, N_n)$ is a permutation of $(1, \ldots, n)$. Let p_i be the probability that the seat number of the first passenger 1 is i, i.e.,

$$p_i = \mathrm{P}(N_1 = i) \quad \text{for } i \in \{1, \ldots, n\}. \tag{2}$$

Then

$$\sum_{i=1}^{n} p_i = 1, \tag{3}$$

and the assumptions of (I) and (II) are expressed as

$$\begin{cases} \text{(I)} & p_1 = \ldots = p_n = \frac{1}{n}, \\ \text{(II)} & p_1 = 0 \text{ and } p_2 = \ldots = p_n = \frac{1}{n-1}, \end{cases} \tag{4}$$

respectively. Let A_l be the event that the passenger l sits in his or her assigned seat, namely,

$$A_l = \{N_l = l\} \quad \text{for } l \in \{1, \ldots, n\}. \tag{5}$$

Since many authors investigate (I) (see [3], [4], [5]), we briefly explain some results for (I). Both [4, (1)] and [5] state that

$$\text{if (I) of (4) holds then } \mathrm{P}(A_l) = \frac{n-l+1}{n-l+2} \quad \text{for } l \in \{2, \ldots, n\}, \tag{6}$$

in particular $\mathrm{P}(A_n) = \frac{1}{2}$. Bollobás [3, p.177] proves it without using mathematical expressions. Moreover Henze and Last [4, Theorem 1] show that $A_2, \ldots, A_n$ are independent, but a simpler proof is given by [1, Theorem 1].

In this Article, we study this problem when the first passenger randomly chooses a seat in the sense of (2). Throughout this Article, we assume

$$p_k > 0 \quad \text{for } k \in \{2, \ldots, n-1\}, \tag{7}$$

which includes (4), since $p_1 = 0$ or $p_n = 0$ is allowed. Under (7), we establish a necessary [as well as sufficient] condition on $p_1, \ldots, p_n$ for the independence of $A_2, \ldots, A_n$ as follows.

Theorem 1: Suppose that $n \geq 3$ and the first passenger chooses his or her seat with probability $p_1, \ldots, p_n$ satisfying (7). Then we have

$$p_1 = p_3 = \ldots = p_n \text{ if and only if } A_2, \ldots, A_n \text{ are independent.} \tag{8}$$

Note that the following example shows that the natural condition (I) of (4) above, i.e. $p_1 = p_2 = \ldots = p_n$, is *not* necessary.

Example 1: Let $n = 3$, and if $p_1 = p_3$ then simple calculations show that we have $\mathrm{P}(A_2) = 2p_1, \mathrm{P}(A_3) = 1/2$ and $\mathrm{P}(A_2 \cap A_3) = p_1$, which gives $\mathrm{P}(A_2 \cap A_3) = \mathrm{P}(A_2)\mathrm{P}(A_3)$. Hence A_2 and A_3 may be independent even if $p_1 = p_2 = p_3 = 1/3$ fails.

This Article is organised as follows. Section 2 provides preliminary results for Theorem 1. We prove Theorem 1 in Section 3, and make concluding remarks in Section 4.

2. *Preliminary results*

Let us introduce notation for the conditional probabilities

$$\alpha_k(l) = \mathrm{P}(A_l|N_1 = k) \quad \text{for } k \in \{2, \ldots, n-1\} \text{ and } l \in \{1, 2, \ldots, n\},$$

which are well-defined because of (7). When the first passenger sits in a seat k for $k \in \{2, \ldots, n-1\}$, the following lemma holds.

Lemma 1: For $k \in \{2, \ldots, n-1\}$, we obtain

$$\begin{cases} \alpha_k(1) = \alpha_k(k) = 0, \\ \alpha_k(l) = 1 \text{ for } l \in \{2, \ldots, k-1\} \text{ with } k \geq 3, \end{cases} \tag{9}$$

and

$$\alpha_k(l) = \frac{n-l+1}{n-l+2} \quad \text{for } l \in \{k+1, \ldots, n\}. \tag{10}$$

Proof: Let us fix $k \in \{2, \ldots, n-1\}$. From the statement of the problem, (9) follows. For simplicity we set $\mathrm{P}_k(\cdot) = \mathrm{P}(\cdot|N_1 = k)$. Since the passenger k randomly chooses a seat in $\{1\} \cup \{k+1, \ldots, n\}$, it turns out that

$$\mathrm{P}_k(N_k = i) = \frac{1}{n-k+1} \quad \text{for} \quad i \in \{1\} \cup \{k+1, \ldots, n\}. \tag{11}$$

When $k \in \{2, \ldots, n-2\}$ we have

$$\mathrm{P}_k(A_l|N_k = i) = \alpha_i(l) \quad \text{for} \quad i \in \{k+1, \ldots, n-1\} \text{ and } l \in \{i+1, \ldots, n\}, \tag{12}$$

and when $k = n-1$ we have

$$\mathrm{P}_{n-1}(A_n|N_{n-1} = n) = 0, \quad \mathrm{P}_{n-1}(A_n|N_{n-1} = 1) = 1. \tag{13}$$

Moreover

$$\begin{cases} \mathrm{P}_k(A_l|N_k = 1) = 1 & \text{for } l \in \{k+1, k+2, \ldots, n\}, \\ \mathrm{P}_k(A_l|N_k = i) = 1 & \text{for} \begin{cases} k \in \{2, \ldots, n-2\}, \\ i \in \{k+2, \ldots, n\}, \\ l \in \{k+1, k+2, \ldots, i-1\}, \end{cases} \\ \mathrm{P}_k(A_l|N_k = l) = 0 & \text{for } l \in \{k+1, k+2, \ldots, n\}. \end{cases} \tag{14}$$

Then it follows that for $k \in \{2, \ldots, n-2\}$ and $l \in \{k+1, k+2, \ldots, n\}$

$$\begin{aligned}
\alpha_k(l) &= \mathrm{P}_k(A_l) = \sum_{i \in \{1\} \cup \{k+1, \ldots, n\}} \mathrm{P}_k(A_l | N_k = i) \mathrm{P}_k(N_k = i) \\
&= \frac{1}{n-k+1} \Big\{ \mathrm{P}_k(A_l | N_k = 1) + \sum_{i=k+1}^{l-1} \mathrm{P}_k(A_l | N_k = i) \\
&\qquad + \sum_{i=l+1}^{n} \mathrm{P}_k(A_l | N_k = i) \Big\} \\
&\overset{(12),(14)}{=} \begin{cases} \frac{n-k}{n-k+1} & \text{if } l = k+1, \\ \frac{n-l+1+\sum_{i=k+1}^{l-1} \alpha_i(l)}{n-k+1} & \text{if } l \in \{k+2, \ldots, n\}. \end{cases} \qquad (15)
\end{aligned}$$

Although solving this equation under (9) yields (10), we prove it by induction with k as in [2, 1.8.39, p.197]. If $k = n-1$ then

$$\begin{aligned}
\alpha_{n-1}(n) &= \mathrm{P}_{n-1}(A_n) = \mathrm{P}_{n-1}(A_n | N_{n-1} = 1) \mathrm{P}_{n-1}(N_{n-1} = 1) \\
&\quad + \mathrm{P}_{n-1}(A_n | N_{n-1} = n) \mathrm{P}_{n-1}(N_{n-1} = n) \overset{(11),(13)}{=} \frac{1}{2}.
\end{aligned}$$

Next, we suppose that (10) is true for $k \in \{n-j, \ldots, n-1\}$. Then we check (10) with $k = n-j-1 \geq 2$. If $l = n-j$ then $\alpha_{n-j-1}(n-j) = \frac{j+1}{j+2}$ from (15). If $l \in \{n-j+1, \ldots, n\}$ then we have

$$\alpha_{n-j-1}(l) = \frac{n-l+1+\sum_{i=n-j}^{l-1} \alpha_i(l)}{n-(n-j-1)+1} = \frac{n-l+1}{n-l+2}.$$

Hence we obtain (10), which completes the proof of Lemma 1.

Remark 1:

(i) Equation (10) with $l = n$ implies $\alpha_k(n) = \frac{1}{2}$ for $k \in \{2, \ldots, n-1\}$. This suggests that if the first passenger sits in a seat $k \in \{2, \ldots, n-1\}$ then the seats 1 and n are chosen with the same probability.

(ii) Equation (12) means that whether the first passenger or the passenger k sits in the seat i, the conditional probability for the passenger l does not change. We use this *memoryless property* when proving the independence of $A_2, \ldots, A_n$ in Theorem 1.

Proposition 1: Make the same assumption of Theorem 1. Then the probability that the passenger l sits in his or her assigned seat is

$$\mathrm{P}(A_l) = \begin{cases} p_1 & \text{for } l = 1, \\ 1 - p_2 & \text{for } l = 2, \\ 1 - \frac{1}{n-l+2}\sum_{k=2}^{l-1} p_k - p_l & \text{for } l \in \{3, \dots, n\}. \end{cases} \tag{16}$$

Proof: If $l = 1$ then $\mathrm{P}(A_1) = \mathrm{P}(N_1 = 1) = p_1$. Let us assume $l \in \{2, \dots, n-1\}$. Conditioned by N_1, we have

$$\begin{aligned} \mathrm{P}(A_l) &= \mathrm{P}(A_l \cap \{N_1 = 1\}) + \sum_{k=2}^{n-1} \mathrm{P}(A_l | N_1 = k)\mathrm{P}(N_1 = k) + \mathrm{P}(A_l \cap \{N_1 = n\}) \\ &= p_1 + \sum_{k=2}^{n-1} \alpha_k(l) p_k + p_n. \end{aligned}$$

Lemma 1 implies the following.

- If $l = 2$ then $\mathrm{P}(A_2) = p_1 + \sum_{k=3}^{n} p_k = 1 - p_2$.
- If $l \in \{3, \dots, n-1\}$ then

$$\begin{aligned} \mathrm{P}(A_l) &= p_1 + \sum_{k=2}^{l-1} \alpha_k(l) p_k + \sum_{k=l+1}^{n-1} \alpha_k(l) p_k + p_n \\ &\overset{(10)}{=} p_1 + \frac{n-l+1}{n-l+2} \sum_{k=2}^{l-1} p_k + \sum_{k=l+1}^{n} p_k \overset{(3)}{=} 1 - \frac{1}{n-l+2} \sum_{k=2}^{l-1} p_k - p_l. \end{aligned}$$

Finally, if $l = n$ then $\mathrm{P}(A_n) = p_1 + \sum_{k=2}^{n-1} \alpha_k(n) p_k \overset{(10),(3)}{=} 1 - \frac{1}{2}\sum_{k=2}^{n-1} p_k - p_n$. Hence (16) holds, which completes the proof.

Remark 2: Proposition 1 tells us that for $l \in \{2, \dots, n\}$ the probability $\mathrm{P}(A_l)$ depends only on $p_2, \dots, p_l$, and is smaller than $1 - p_l$, which is the probability that the first passenger sits except the seat l, by $\frac{1}{n-l+2}\sum_{k=2}^{l-1} p_k$. In addition it implies that

$$p_1 = p_n \text{ if and only if } \mathrm{P}(A_n) = \frac{1}{2}. \tag{17}$$

In fact, combining (16) with $l = n$ and (3) yields $\mathrm{P}(A_n) = \frac{1+p_1-p_n}{2}$, which gives (17). Note that (17) corresponds to Remark 1 (i).

Example 2:

- *Case* (I): Equation (16) with $p_1 = p_2 = \ldots = p_n = \frac{1}{n}$ implies (6).
- *Case* (II): Equation (16) with $p_1 = 0$ and $p_2 = \ldots = p_n = \frac{1}{n-1}$ implies

$$\mathrm{P}(A_l) = \frac{n-l+1}{n-l+2} - \frac{1}{(n-1)(n-l+2)} \quad \text{for } l \in \{2, \ldots, n\},$$

  whose form suggests the difference from (6).

We remark that (1) follows from Cases (I) and (II) with $l = n$, respectively.

3. *Proof of Theorem* 1

Suppose $p_1 = p_3 = \ldots = p_n$. Then we show

$$\mathrm{P}(A_j|A_i^c) = \mathrm{P}(A_j) \quad \text{for } 2 \le i < j \le n, \tag{18}$$

noting that $\mathrm{P}(A_j|A_i^c)$ is well-defined since $\mathrm{P}(A_i^c) \ge p_i > 0$ for $i \in \{2, \ldots, n-1\}$. It follows that

$$\mathrm{P}(A_j|A_i^c) = \mathrm{P}(A_j|N_1 = i) = \alpha_i(j) \overset{(10)}{=} \frac{n-j+1}{n-j+2}, \tag{19}$$

where the first equality holds for the same reason as (12). Using (16) and

$$p_2 = 1 - (n-1)p_1, \tag{20}$$

we have

$$\mathrm{P}(A_j) = \frac{n-j+1}{n-j+2} \quad \text{for } j \in \{3, \ldots, n\}, \tag{21}$$

because

- if $j \in \{3, \ldots, n-1\}$ then $\mathrm{P}(A_j) \overset{(16)}{=} 1 - \frac{p_2+(j-3)p_1}{n-j+2} - p_1 \overset{(20)}{=} \frac{n-j+1}{n-j+2}$,
- if $j = n$ then $\mathrm{P}(A_n) \overset{(17)}{=} \frac{1}{2}$.

Therefore (18) holds, which implies that A_i and A_j are independent by using [2, 1.5.1, p.3]. Similarly, to show that $A_2, \ldots, A_n$ are independent, it is sufficient to prove for any $m \in \{2, 3, \ldots, n-2\}$ and $2 \leq j_0 < j_1 < j_2 < \ldots < j_m \leq n$,

$$\mathrm{P}\left(\bigcap_{s=1}^{m} A_{j_s}^c \;\middle|\; A_{j_0}^c\right) = \prod_{s=1}^{m} \mathrm{P}\left(A_{j_s}^c\right), \tag{22}$$

which follows from

$$\begin{aligned}
&\text{LHS of (22)} \stackrel{(12)}{=} \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m} A_{j_s}^c\right) = \mathrm{P}_{j_0}\left(A_{j_m}^c \;\middle|\; \bigcap_{s=1}^{m-1} A_{j_s}^c\right) \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m-1} A_{j_s}^c\right) \\
\stackrel{(12)}{=} \;& \mathrm{P}_{j_{m-1}}\left(A_{j_m}^c\right) \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m-1} A_{j_s}^c\right) = \{1 - \alpha_{j_{m-1}}(j_m)\} \mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m-1} A_{j_s}^c\right) \\
= \;& \prod_{s=1}^{m} \{1 - \alpha_{j_{s-1}}(j_s)\} \stackrel{(10)}{=} \prod_{s=1}^{m} \frac{1}{n - j_s + 2} \stackrel{(21)}{=} \text{RHS of (22)}.
\end{aligned}$$

Note that $\mathrm{P}_{j_0}\left(A_{j_m}^c \mid \bigcap_{s=1}^{m-1} A_{j_s}^c\right)$ is also well-defined because it turns out that $\mathrm{P}_{j_0}\left(\bigcap_{s=1}^{m-1} A_{j_s}^c\right) > 0$ from (7). Hence $A_2, \ldots, A_n$ are independent.

Next, we suppose that $A_2, \ldots, A_n$ are independent. Then (21) is obtained by (18) and (19). Hence (16) yields $1 - \frac{1}{n-l+2} \sum_{k=2}^{l-1} p_k - p_l = \frac{n-l+1}{n-l+2}$, so that

$$p_l = \frac{p_1 + p_l + \ldots + p_n}{n - l + 2} \quad \text{for } l \in \{3, \ldots, n\}.$$

If $l = n$ then $p_1 = p_n$. If $l = n-1$ then $p_{n-1} = \frac{p_1 + p_{n-1} + p_n}{3}$, which implies $p_{n-1} = p_1 = p_n$. Repeating this procedure leads to $p_1 = p_3 = \ldots = p_n$, which completes the proof.

4. *Conclusion*

Let us remark that the condition (7) is required for Theorem 1. Indeed, if (7) is violated then $A_2, \ldots, A_n$ are independent for $p_1 = 1$ or $p_n = 1$ which does not satisfy $p_1 = p_3 = \ldots = p_n$.

Finally, it would be interesting to have an intuitively clear reason why the value of p_2 is independent of the result of Theorem 1.

Acknowledgements

The authors would like to thank the referee for helpful suggestions that have improved this Article. This research was supported by KAKENHI 19K03622 of Japan Society for the Promotion of Science.

References

1. G. Grimmett, D. Stirzaker, The lost boarding pass and other practical problems, *Math. Gaz.* **105** (2021) pp. 216-221. `https://doi.org/10.1017/mag.2021.49`

2. G. Grimmett, D. Stirzaker, *One Thousand Exercises in Probability* (3rd edn.) Oxford UP. (2020).

3. B. Bollobás, *The art of mathematics*, Cambridge UP. (2006).

4. N. Henze, G. Last, Absent-Minded Passengers, *Amer. Math. Monthly*, **126** (2019) pp. 867-875. `https://doi.org/10.1080/00029890.2019.1656024`

5. Y. Nigussie, Finding your seat versus tossing a coin, *Amer. Math. Monthly* **121** (2014) pp. 545-546. `https://doi.org/10.4169/amer.math.monthly.121.06.545`

SHOHEI KUBO
TOSHIO NAKATA
NAOKI SHIRAISHI
Department of Mathematics, University of Teacher Education Fukuoka, Munakata, Fukuoka, 811-4192, Japan
e-mail: *nakata@fukuoka-edu.ac.jp*